CDI - II - Prática 24/5/21

Ficha 11 + Ficha 12

- Conservativo ≡ gradiente

$F: \mathbb{R}^n \to \mathbb{R}^n$ $\quad F = \nabla \varphi \quad$ $\varphi: \mathbb{R}^n \to \mathbb{R}, C^1$

- Se $F = \nabla\varphi$ então F é fechado

$$\frac{\partial F_j}{\partial x_k} = \frac{\partial F_k}{\partial x_j}, \; j \neq k$$

- F é conservativo sse $\int_L F = 0$, $\forall_L$ linha fechada.

- T.F.C: Se $F = \nabla\varphi \Rightarrow \int_L F = \varphi(B) - \varphi(A)$

3-a) $a(x,y)$ não é fechado

$\Rightarrow$ a não é conservativo.

$$3x^2 = \frac{\partial a_2}{\partial x} \neq \frac{\partial a_1}{\partial y} = 2y$$

3-b) $b(x,y) = (x^3, y^2) + (y, x)$

$$\varphi(x,y) = \frac{x^4}{4} + \frac{y^3}{3} + xy + C$$

$C \in \mathbb{R}$.

✓

3-c) $\varphi(x,y) = e^x + e^y + C$

$\nabla \varphi = c$ ✓

3-d) $\ell(x,y) = \frac{1}{2}\left(\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}\right)$

$$\varphi(x,y) = \frac{1}{2}\log(x^2+y^2) + C$$

$$\varphi(x,y) = \sqrt{\log(x^2+y^2)} + C \, .$$

3-e) $\ell(x,y,z) = (y, x, 2z)$

$$= (y,x,0) + (0,0,2z)$$

$$\varphi(x,y,z) = xy + z^2 + C$$

$$\underset{\substack{\parallel \\ 1}}{\frac{\partial \ell_2}{\partial x}} = \underset{\substack{\parallel \\ 1}}{\frac{\partial \ell_1}{\partial y}} \; ; \; \underset{\substack{\parallel \\ 0}}{\frac{\partial \ell_3}{\partial x}} = \underset{\substack{\parallel \\ 0}}{\frac{\partial \ell_1}{\partial z}} \; ; \; \underset{\substack{\parallel \\ 0}}{\frac{\partial \ell_2}{\partial z}} = \underset{\substack{\parallel \\ 0}}{\frac{\partial \ell_3}{\partial y}}$$

$$\left\{\begin{array}{l} \dfrac{\partial \varphi}{\partial x} = y \longrightarrow \varphi(x,y,z) = xy + \underbrace{A(y,z)}_{B(z)} \\[2ex] \dfrac{\partial \varphi}{\partial y} = x \longrightarrow \cancel{x} + \dfrac{\partial A}{\partial y} = \cancel{x} \longrightarrow \dfrac{\partial A}{\partial y} = 0 \\[2ex] \dfrac{\partial \varphi}{\partial z} = 2z \end{array}\right.$$

$$A(y,z) = B(z)$$

$$\longrightarrow B'(z) = 2z$$

$$B(z) = z^2 + C$$

$$\boxed{\varphi(x,y,z) = xy + z^2 + C}$$

3-f) não é fechado

$$1 = \frac{\partial f_2}{\partial x} \neq \frac{\partial f_1}{\partial y} = -1$$

não é gradiente!

4 – $F(x,y,z) = \frac{1}{2}\left(\frac{2x}{1+x^2+y^2}, \frac{2y}{1+x^2+y^2}, 4z \right)$

$$\varphi(x,y,z) = \frac{1}{2}\log(1+x^2+y^2) + \underbrace{\frac{1}{2}4\frac{z^2}{2}}_{z^2} + C$$

$$\nabla\varphi = F.$$

TFC: $\displaystyle\int_L F = \varphi(B) - \varphi(A)$

a) $A \rightarrow t=0 \quad A=(1,0,0)$

$$t = 2\pi \longrightarrow B = (1, 0, 2\pi)$$

$$\int_L F = \varphi(1,0,2\pi) - \varphi(1,0,0)$$

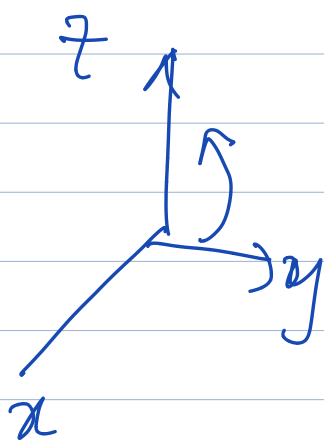
$$= \frac{\log 2}{2} + 4\pi^2 - \frac{\log 2}{2} = 4\pi^2.$$

$$g(t) = (\cos t, \operatorname{sen} t, t)$$

$$A = g(0), \qquad B = g(2\pi).$$

——— // ———

4-b) $\begin{cases} y^2 + z^2 = 1 \\ x = y^2 - z^2 \end{cases}$



$y = \cos t, \quad z = \operatorname{sen} t \qquad x = \cos^2 t - \operatorname{sen}^2 t$

$$0 \le t \le 2\pi$$

$$g(t) = (\cos^2 t - \operatorname{sen}^2 t,\ \cos t,\ \operatorname{sen} t)$$

$$g(0) = (1, 1, 0) = A$$

$$\shortparallel$$

$$g(2\pi) = (1, 1, 0) = B$$

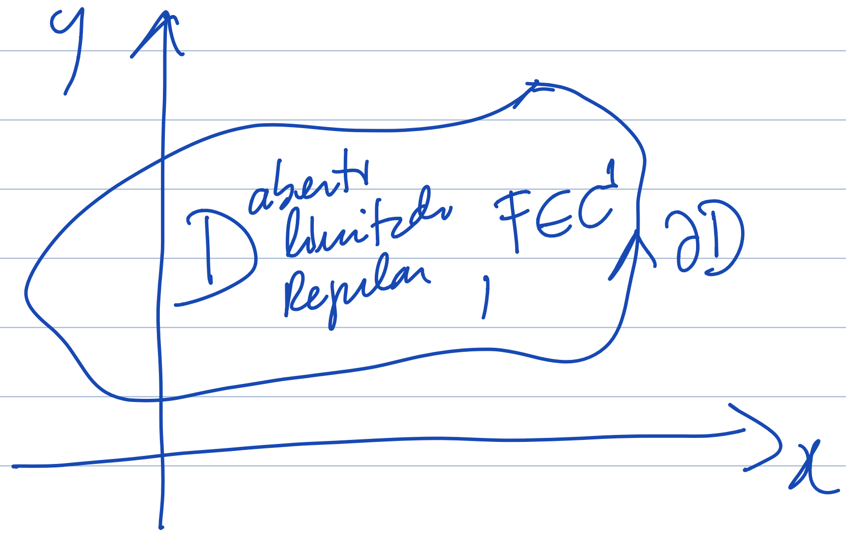
$$\int_L F = 0 \ !$$

Ficha 12 → Teorema de Green.

T. Green: ($\mathbb{R}^2$)

$F: \mathbb{R}^2 \to \mathbb{R}^2$, C^1 em D

$F = (P, Q)$

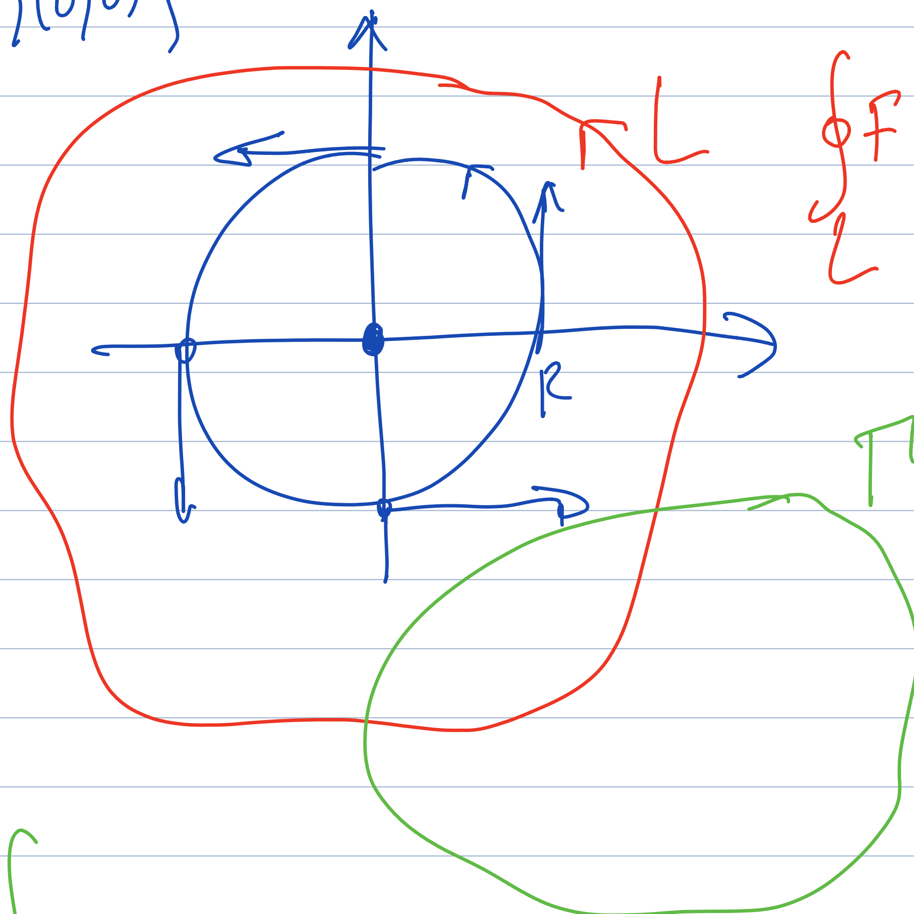


$$\underbrace{\int_{\partial D} F}_{\text{Trabalho de F em } \partial D} = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}\, dx\,dy$$

Palo!

$$F(x,y) = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$$

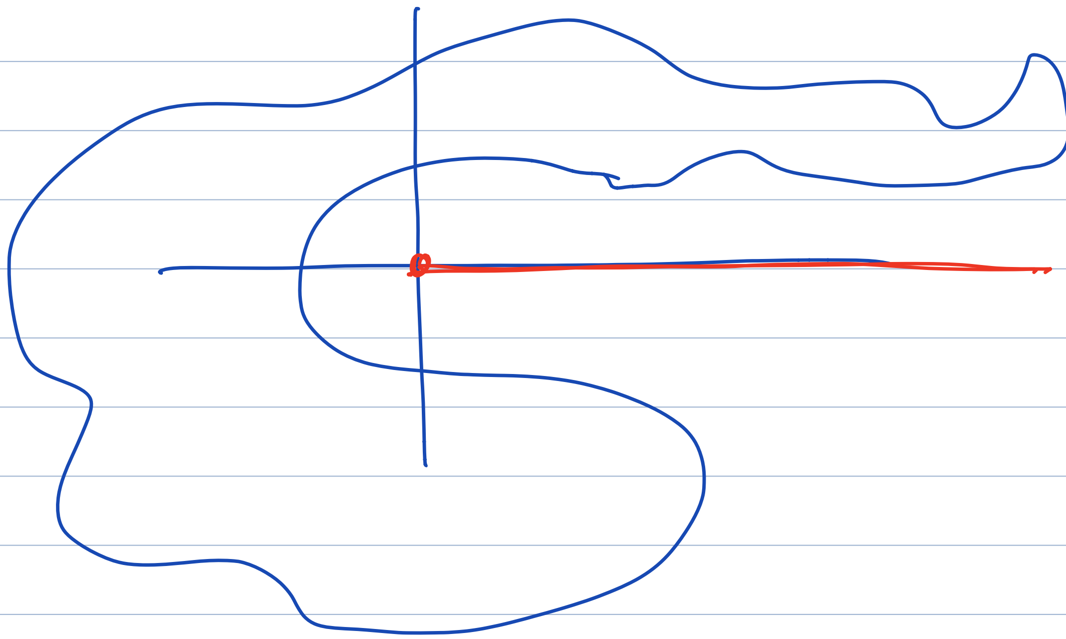
$\mathbb{R}^2 \setminus \{(0,0)\}$



$$\oint_L F = 2\pi$$

$$\oint_\Gamma F = 0$$

$\mathbb{R}^2 \setminus \{(x,0) : x \geq 0\}$

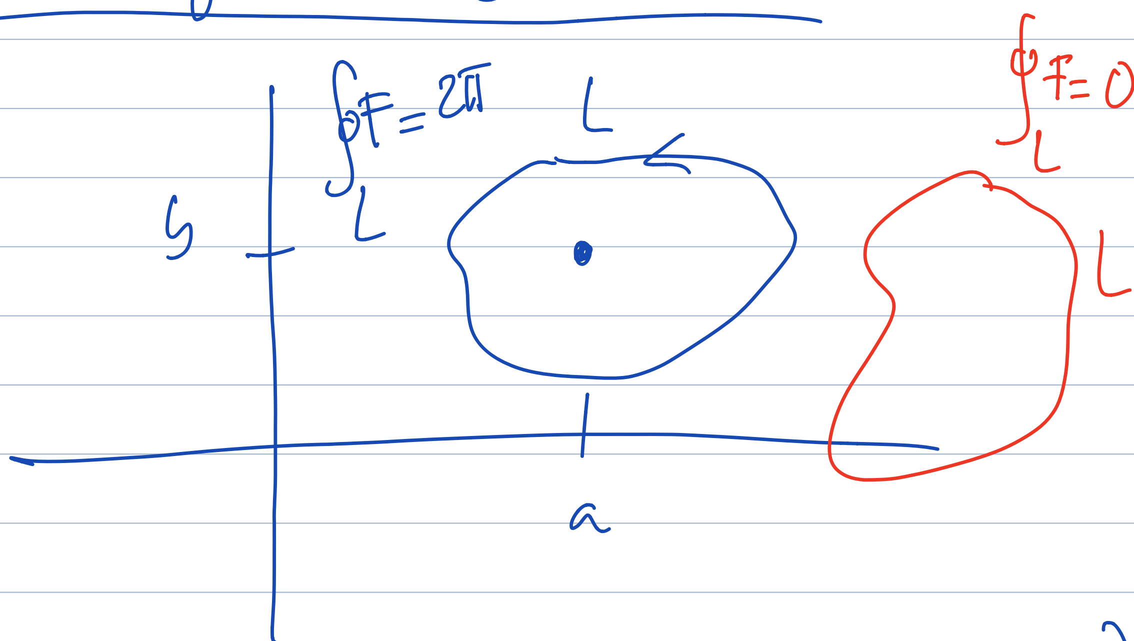


Exercício: $\varphi(x,y) = \arctan \frac{y}{x}$, $x \neq 0$

$$\nabla \varphi(x,y) = R(x,y)$$

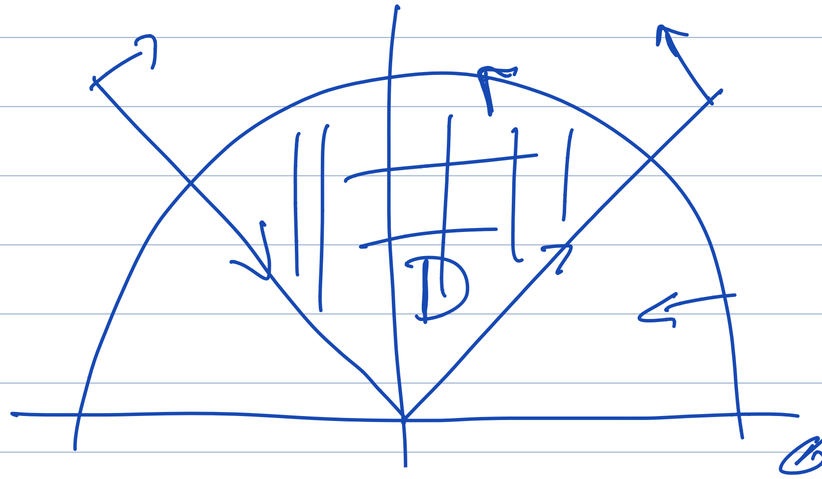
Mas φ e R não têm o mesmo domínio $\mathbb{R}^2 \setminus \{(0,0)\}$.

Caso geral (Ralo):



$$F(x,y) = \left(-\frac{y-b}{(x-a)^2+(y-b)^2}, \frac{x-a}{(x-a)^2+(y-b)^2} \right)$$

1 — $F(x,y) = (P(x,y), Q(x,y)) = (-2y, x)$



$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3$$

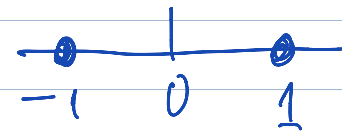
$$\oint_{\partial D} F = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$= \iint_D 3\,dx\,dy = 3\,\mathrm{vol}_2(D)$$

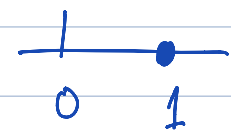
$$= \frac{3}{4}\pi \quad //$$

2- Soma de 2 "ralos".

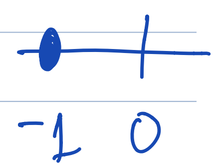
$$F = G + H$$



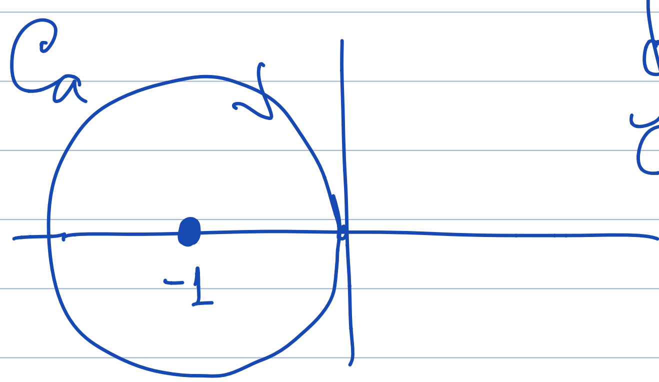
$$G(x,y) = \left(\frac{-y}{(x-1)^2+y^2} , \frac{x-1}{(x-1)^2+y^2} \right)$$



$$H(x,y) = \left(\frac{y}{(x+1)^2+y^2} , \frac{-(x+1)}{(x+1)^2+y^2} \right)$$



a) $(x+1)^2 + y^2 = 1$

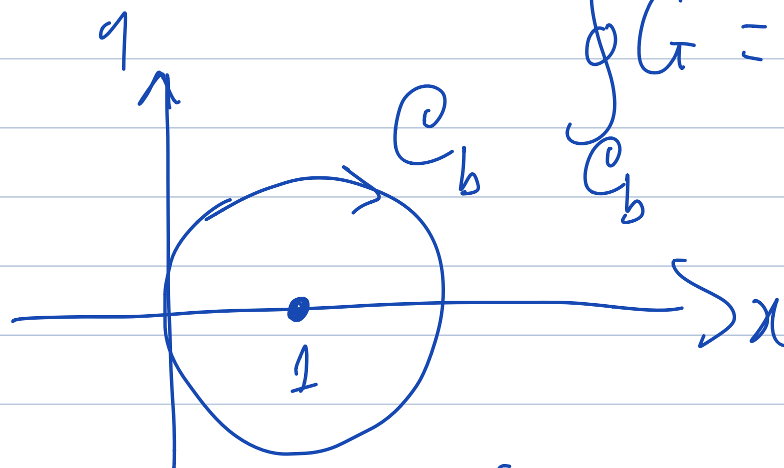


$$\oint_{C_a} G = 0$$

$$\oint_{C_a} H = 2\pi$$

$$\oint_{C_a} F = 0 + 2\pi = 2\pi$$

2-b)

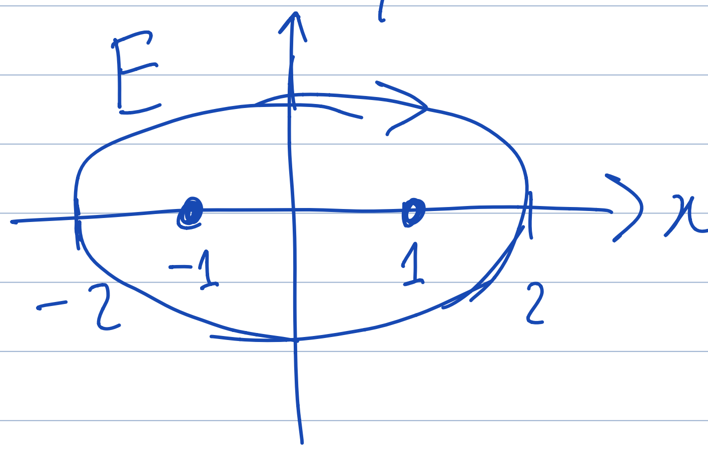


$$\oint_{C_b} G = -2\pi$$

$$\oint_{C_b} H = 0$$

$$\oint_{C_2} F = -2\pi + 0 = -2\pi$$

2-c) $E \quad \frac{x^2}{4} + y^2 = 1$ ↺

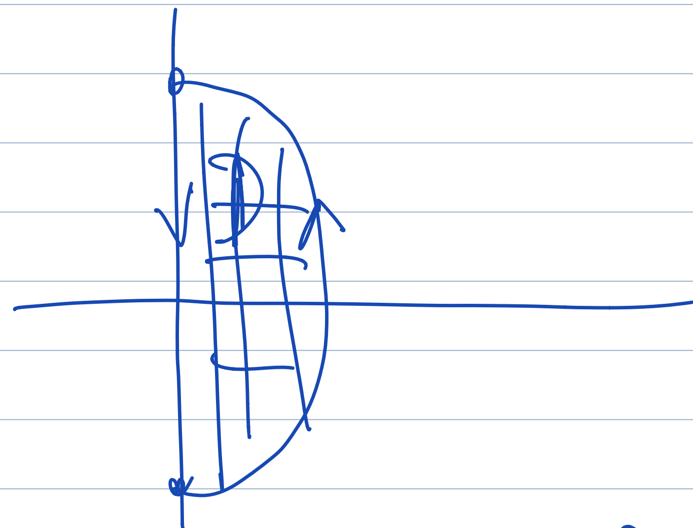


$$\oint_E G = -2\pi$$

$$\oint_E H = 2\pi$$

$$\oint_E F = -2\pi + 2\pi = 0 \quad //$$

3- D: $x^2 + \frac{y^2}{4} < 1$, $x > 0$



$$\iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{Constante } C} dx\,dy = \oint_{\partial D} F$$

$$\| $$

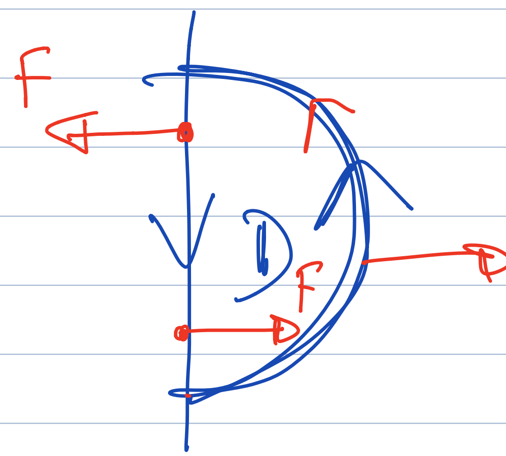
$$C \,\mathrm{Vol}_2(D)$$

"Construir F tal que $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = C$

1) $F(x,y) = (-y, x)$ $\qquad c = 2$

2) $\boxed{F(x,y) = (-y, 0)}$ $\qquad c = 1$

3) $F(x,y) = (0, x)$ $\qquad c = 1$



$g(t) = (\cos t, 2 \operatorname{sen} t)$

$-\frac{\pi}{2} < t < \frac{\pi}{2}$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F(-2\operatorname{sen} t, 0) \cdot (-\operatorname{sen} t, 2\cos t)\, dt$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \operatorname{sen}^2 t\, dt \quad \text{etc} \ldots$$